SOLUTIONS OF PROBLEMS IN NUMBER ONE.

Solutions of problems in number one have been received as follows:—

From Prof. L. G. Barbour, 335, 337; Prof. W. P. Casey, 332, 333; G. M. Day, 335; Geo. Eastwood, 336; Wm. Hoover, 332, 333, 335; A. Hall (son of Prof. A. Hall), 335; H. Heaton, 333; W. E. Heal, 332, 333, 334, 335; Prof. J. H. Kershner, 334; Prof. D. J. Mc Adam, 335, 336; Prof. J. Scheffer, 333, 335; Prof. E. B. Seitz, 333, 334, 335.

332. "Find all the values of x and y in the following equations:

$$x + xy^3 = 18, \qquad (1)$$
$$xy + xy^2 = 12." \qquad (2)$$

SOLUTION BY W. E. HEAL MARION, INDIANA.

Multiply the first equation by 2, the second by 3, subtract and divide by x and we have

$$2(y^3+1)—3y(y+1) = 2(y+1)(y^2—y+1)—3y(y+1)$$
$$= (y+1)(2y^2—5y+1) = 0.$$

Therefore $y+1 = 0$ and $2y^2—5y+1 = 0$; therefore

$$y = —1,\ 2,\ \tfrac{1}{2};$$
$$x = \infty,\ 2,\ 16.$$

333. "Find the value to x terms of the continued fraction

$$\cfrac{2}{1+\cfrac{2}{1+\&c."}}$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $u_x \div v_x$, $u_{x+1} \div v_{x+1}$, $u_{x+2} \div v_{x+2}$ be the xth, $(x + 1)$th, $(x + 2)$th converging fractions. Then

$$\frac{u_{x+1}}{v_{x+1}} = 2\div\left(1+\frac{u_x}{v_x}\right) = \frac{2v_x}{u_x + v_x},$$

whence $u_{x+1} = 2v_x$ (1), $v_{x+1} = u_x+v_x$; and simil'y $v_{x+2} = u_{x+1}+v_{x+1}$. (2)

From (1) and (2) we find $v_{x+2} = v_{x+1} + 2v_x$, an equation in finite differences whose solution gives $v_x = C_1(2)^x + C_2(—1)^x$. (3)

When $x = 1$, $v_1 = 2C_1—C_2 = 1$, and when $x = 2$, $v_2 = 4C_1 + C_2 = 3$; whence $C_1 = \frac{2}{3}$, and $C_2 = \frac{1}{3}$; $\therefore v_x = \frac{1}{3}(2^{x+1} \pm 1)$, $u_x = 2v_{x-1} = \frac{1}{3}(2^{x+1} \mp 2)$, and

$$\frac{u_x}{v_x} = \frac{2^{x+1} \mp 2}{2^{x+1} \pm 1},$$

the upper sign being taken when x is even, and the lower when x is odd.

334. "Pairs of tangents which meet always at the same angle are drawn to a given ellipse. Find the envelope of the chords of contact."

SOLUTION BY W. E. HEAL.

Let $(x^2 \div a^2)+(y^2 \div b^2) = 1$ be the equation of the given ellipse; $(a \cos \theta, b \sin \theta)$, $(a \cos \varphi, b \sin \varphi)$ the coordinates of the points of contact; α the constant angle at which the tangents intersect.

Then the equations of the tangents and chord of contact are

$$(x \div a) \cos \theta+(y \div b) \sin \theta = 1, \tag{1}$$

$$(x \div a) \cos \varphi+(y \div b) \sin \varphi = 1, \tag{2}$$

$$(x \div a) \cos \tfrac{1}{2}(\varphi+\theta)+(y \div b) \sin \tfrac{1}{2}(\varphi+\theta) = \cos \tfrac{1}{2}(\varphi-\theta). \tag{3}$$

By a well known formula the angle between the tangents is

$$\tan \alpha = \frac{\sin \theta \cos \varphi - \cos \theta \sin \varphi}{\sin \theta \sin \varphi + \cos \theta \cos \varphi} = \tan (\theta - \varphi). \tag{4}$$

$$\therefore \alpha = n\pi + (\theta - \varphi). \tag{5}$$

$$\therefore \varphi = (n\pi - \alpha) + \theta = 2[\tfrac{1}{2}(n\pi - \alpha)]+\theta = 2\beta+\theta. \tag{6}$$

Substituting in (3) we find for the equation of the chord of contact

$$(x \div a) \cos (\theta+\beta)+(y \div b) \sin (\theta+\beta) = \cos \beta. \tag{7}$$

Equating to zero the differential of (7) with respect to θ we have

$$(y \div b) \cos (\theta+\beta)-(x \div a) \sin (\theta+\beta) = 0. \tag{8}$$

The sum of the squres of (7) and (8) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \beta. \tag{9}$$

Equation (9) is the required envelope which is, therefore, an ellipse.

---

335. "The curve whose rectangular equation is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = r^{\frac{1}{2}}$ revolves around the axis of x. Determine the volume of the solid thus described between the limits $x = 0$ and $x = r$."

SOLNTION BY A. HALL, HARVARD COLLEGE.

The differential of the volume is

$$dV = \pi y^2 \, dx;$$

and, substituting the value of y^2 from the given equation and integrating, we have

$$V = \pi\left(r^2 x+\frac{x^3}{3}+3rx^2 - \frac{8}{3} r^{\frac{3}{2}} x^{\frac{3}{2}} - \frac{8}{5} r^{\frac{1}{2}} x^{\frac{5}{2}} \right).$$

Taking the limits and reducing,

$$V = \frac{\pi r^3}{15}.$$

336. "In a locomotive engine there are given: The impressed force of the steam on the piston, the radius of the crank, and the length of the connecting rod: To find the uniform force which, if applied at right angles to the end of the crank, would do the same work as the impressed force."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

Let F = the area of the piston; S = distance traveled by piston before expansion begins; that is, before steam is cut off; r = length of crank; S_1 = entire length of stroke = $2r$. And let p = steam pressure per unit of area before expansion, expressed in atmospheres; p_1 = steam pressure per square unit of area after expansion; q = back pressure per unit, P = uniform pressure on end of crank which we are seeking.

Then, assuming expansion according to Mariotte's Law, the complete work of the steam per stroke (Weisbach, 2nd vol., Part second, p. 366) is

$$A = FS_1p_1\left[1+\log\left(\frac{S_1}{S}\right)-\frac{q}{p_1}\right] = S_1P_1, \text{ say,}$$

in which $p_1 = pS \div S_1$ and $q = 1$ for this non-condensing engine.

Now, by principle of virtual moments $P\pi r = S_1P_1 = 2rP_1$. Therefore $P = 2P_1 \div \pi$. That is, the uuiform force is $2 \div \pi$ of the average pressure on the piston.

---

337. "Required the constant quantity into which if we divide the periodic time of any planet, multiplied by its third root, the quotient will be the distance such planet falls from a tangent to its orbit in one second of time: i. e., solve the equation,

$$\frac{\text{Constant quantity}}{(\text{Periodic time})^{\frac{4}{3}}} = \text{Fall from tang.}$$

SOLUTION BY PROF. L. G. BARBOUR, RICHMOND, KY.

Let s = space any planet falls through in 1 second; $F = 2s$ = force of gravity, t being 1 second; T = number of seconds in periodic time of planet; R = radius vector; and let a = semi-major axis of elliptic orbit, then is mean value of $R = a$; and we have

$$s = \tfrac{1}{2}F = \frac{2\pi^2a^3}{T^2R^2} = \frac{2\pi^2a}{T^2} \text{ for mean value of } R.$$

Let $T' = 1$ second, then, by Kepler's third law, $T'^2 : T^2 :: a'^3 : a^3$, or

$$1 : T^2 :: a'^3 : a^3; \therefore T^2 = \frac{a^3}{a'^3}; \; T^{\frac{4}{3}} = \frac{a^2}{a'^2}. \quad \therefore sT^{\frac{4}{3}} = 2\pi^2a',$$

where a' = mean distance of a body revolving about the sun's center in one second, supposing the mass of the sun to be concentrated at its center.